ISRAEL JOURNAL OF MATHEMATICS 98 (1997), 29–60

SUMS OF CONTINUOUS AND DIFFERENTIABLE FUNCTIONS IN DYNAMICAL SYSTEMS*

BY

PIERRE LIARDET

CMI, 39 rue Joliot-Curie
F-13453 Marseille cedex 13, France

AND

DALIBOR VOLNÝ

Matematicky Ustav Univerzity Karlovy, Pri Matematicko Fyzikalni Fakulte UK
18600 Praha 8, Sokolovska 83, Czech Republic

ABSTRACT

Let T be a homeomorphism of a metrizable compact X, the sequence c_k/k tends to 0 and c_k tends to infinity. We'll study the limit behaviour of the distributions of the sums $(1/c_k)\sum_{i=0}^{k-1} F \circ T^i$ where F is from a space of continuous functions — the central limit problem and the speed of convergence in the ergodic theorem.

The main attention is given to the case where X is the unit circle and T is an irrational rotation; in this case we consider the spaces of absolutely continuous, Lipschitz, and k-times differentiable functions F.

1. Introduction

Let $(T, \Omega, \mathcal{A}, \lambda)$ be a dynamical system given by a measure preserving transformation $T\colon \Omega \to \Omega$ of the probability space $(\Omega, \mathcal{A}, \lambda)$. The limit behavior of the sums $S_n(f) = \sum_{i=0}^{n-1} f \circ T^i$ is a topic which is intensively studied in ergodic theory and probability theory. The Birkhoff ergodic theorem says that if f is integrable, $(1/n)S_n(f)$ converge pointwise a.e. Burton and Denker proved that

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* The research of the first author was partially supported by URA-CNRS 225 and the research of the second author was partially supported by the Charles University grant GAUK 368.
Received December 18, 1994

for each ergodic and aperiodic dynamical system there exists f such that for the process $(f \circ T^i)$ the central limit theorem holds ([2], see also [4]). Lacey proved a convergence to self-similar processes ([14]). On the other hand, Volný [20] proved that for a generic set of functions from L_0^p, $1 \leq p < \infty$ and any sequence $(c_n)_n$ such that $\lim_n c_n = \infty$ and $\lim_n c_n/n = 0$, the distributions of $(1/c_n)S_n(f)$ converge (along subsequences) to all probability laws (the case of $p = \infty$ can be proved in the same way, using the fact that measurable coboundaries are dense in L_0^∞, cf. [9]). This shows that in the Birkhoff theorem, an arbitrarily slow rate of convergence is not only possible but also generic (more results on the speed of convergence in the ergodic theorem are given in the monograph [12]).

Here we shall first consider the situation when T is a homeomorphism of a topological space and f is continuous. When considering the space of continuous functions $\mathcal{C}_0(X)$ on a metrizable compact space X, with null integral for each T-invariant Borel probability measure, we get the same results as for the L_0^p spaces on a general probability space with an aperiodic measure preserving transformation (Theorem 1). Further results concern the spaces of absolutely continuous, Lipschitz, and continuously differentiable functions for an irrational rotation of the unit circle (our results thus treat one of the simplest differentiable dynamical systems). The Koksma–Denjoy Theorem shows that for bounded variation functions, the rate of convergence in the Ergodic Theorem cannot be arbitrarily slow. For example, if the rotation is given by the golden number, the suprema of the partial sums of any zero mean absolutely continuous function grow with a rate slower than the logarithmic one. As we shall see, the results depend on Diophantine properties of the rotation.

In the case of rotations $x \mapsto x+\alpha \bmod 1$ where the continued fraction expansion of α has unbounded partial quotients, we shall show results similar to those which hold in L^p spaces or in the space of continuous functions; our problem can be considered as satisfactorily answered in this case. As a corollary we get a (known) result giving conditions for the existence and genericity of k-times continuously differentiable zero mean functions which are not coboundaries. We shall also prove the existence and genericity of ergodic cocycles for a cylindric flow.

The results concerning the bounded partial quotients are much less complete. We shall prove that for a generic set of absolutely continuous zero mean functions F, there exists a weak convergence to the standard normal law along subsequences of $(1/c_n)S_n(F)$, with $\lim_n c_n = \infty$. The partial sums $S_n(F)$ are thus not stochastically bounded, hence F are not coboundaries (see [15]). We shall give bounds for the suprema of $S_n(F)$.

The paper is based on research done during a visit of the first author in Praha and several visits of the second author in Marseille.

2. Continuous functions on a compact metric space

Let X be a compact metric space, $T: X \to X$ a homeomorphism of X onto itself and let λ be a T-invariant Borel probability measure. We suppose that the measure λ is aperiodic (see [5]). $\mathcal{C}(X)$ denotes the space of all real valued continuous functions on X and $\mathcal{C}_0(X)$ is the space of all $f \in \mathcal{C}(X)$ with $\int f\,d\tilde{\lambda} = 0$ for any T-invariant probability measure $\tilde{\lambda}$. By $\|.\|_\infty$ we denote the supremum norm in $\mathcal{C}(X)$.

THEOREM 1: *Let the measure λ be aperiodic. Let $(c_n)_n$ be an unbounded increasing sequence of positive integers with c_n/n converging to 0. Then there exists a dense G_δ set of functions $f \in \mathcal{C}_0(X)$ such that for each probability measure ν on $\mathbb{R}$ there exists a sequence of positive integers $n_k \to \infty$ for which the sequence of distributions of $(1/c_{n_k}) \sum_{j=1}^{n_k} f \circ T^j$ weakly converges to ν.*

As a corollary we get a result saying that for continuous functions, the convergence in the Birkhoff theorem may be arbitrarily slow (cf. [12]).

COROLLARY: *Let $(c_n)_n$ be as in Theorem 1. Then there exists a dense G_δ set of functions $f \in \mathcal{C}_0(X)$ for which there exist subsequences $(c_{n_k})_k$ such that $k \mapsto (1/c_{n_k}) \sum_{j=1}^{n_k} f \circ T^j$ diverges to infinity in the measure.*

Let $U: \mathcal{C}(X) \to \mathcal{C}(X)$ be the operator defined by $Uf = f \circ T$, and let I be the identity operator.

For proving the Theorem we shall need to show that in $\mathcal{C}_0(X)$, the set of coboundaries is dense. Similarly as in the L^p spaces, $0 < p \leq \infty$, this can be easily proved using a duality argument. The results should be well known to the specialists, e.g. the next lemma can be found as Proposition 9.12 in [8] (Professor Lemańczyk kindly informed the authors about this reference.)

LEMMA 1: *$(I - U)\mathcal{C}(X)$ is a dense subset of $\mathcal{C}_0(X)$.*

Proof: Let us suppose that $(I-U)\mathcal{C}(X)$ is not dense in $\mathcal{C}_0(X)$. Then there exists a finite real-valued measure $\tilde{\lambda}$ on X and $f \in \mathcal{C}_0(X)$ such that $\int f\,d\tilde{\lambda} \neq 0$ and

$\int g\, d\tilde{\lambda} = 0$ for all $g \in (I-U)\mathcal{C}(X)$. For $h \in \mathcal{C}(X)$ we thus have $\int (h - Uh)\, d\tilde{\lambda} = 0$, hence $\tilde{\lambda}$ is a T-invariant measure. Let $\{A, B\}$ be a measurable partition of X representing the Hahn decomposition of $\tilde{\lambda}$, i.e. $\tilde{\lambda}(E) \geq 0$ for every measurable set $E \subset A$ and $\tilde{\lambda}(F) \leq 0$ for every measurable set $F \subset B$. Let $C = A \setminus T^{-1}A$. Then $\tilde{\lambda}(C) \geq 0$ while $TC \cap A = \emptyset$, hence $TC \subset B$, hence $\tilde{\lambda}(TC) \leq 0$. From this and $\tilde{\lambda}(C) = \tilde{\lambda}(TC)$ follows $\tilde{\lambda}(C) = 0$, therefore A is an invariant set and B is thus invariant as well. Considering the multiples of the measures $\tilde{\lambda}^+(E) = \tilde{\lambda}(E \cap A)$ and $\tilde{\lambda}^-(E) = \tilde{\lambda}(E \cap B)$ we derive that there exists an invariant probability measure $\tilde{\lambda}_0$ with $\int f\, d\tilde{\lambda}_0 \neq 0$, which contradicts the supposition $f \in \mathcal{C}_0(X)$. ∎

Proof of Theorem 1: The proof can be done in a similar way as the proof of Theorem 1 in [20]; Lemma 1 plays the same role as the Lemma in [20]. Notice that the density of $(I-U)\mathcal{C}(X)$ in $\mathcal{C}_0(X)$ is sufficient, in fact we do not need the density of $(I-U)\mathcal{C}_0(X)$.

Let Γ be the set of all probability measures ν on $\mathbb{R}$ supported by finite sets of rational numbers, with rational values, and $\int x\, d\nu(x) = 0$. One can easily see that Γ is a countable set, dense in the space of all probability distributions on $\mathbb{R}$ (equipped with the topology of weak convergence).

The set of the weak limit points of the distributions of $(1/c_n)S_n(f)$ is closed for each $f \in \mathcal{C}_0(X)$, hence for the proof of Theorem 1 it suffices to prove that for an arbitrarily chosen $\nu \in \Gamma$ there exists a dense G_δ set of $f \in \mathcal{C}_0(X)$ for which the distributions of $(1/c_{n_k})S_{n_k}(f)$ weakly converge to ν for some sequence of positive integers $n_k \to \infty$.

Let $\nu \in \Gamma$ and let n be a positive integer. There exists a positive integer m and rational numbers $x_1, \ldots, x_r$, $r \leq m$, such that $\nu(\{x_j\}) = m(j)/m$, $1 \leq j \leq r$, where $m(j)$ are positive integers such that $\sum_{j=1}^r m(j) = m$ and $\sum_{j=1}^r x_j m(j) = 0$. The measure λ is regular, so that for every measurable set A and $\epsilon > 0$ there exist an open set $A \subset A'$ and a closed set $A'' \subset A$ with $\lambda(A' \setminus A'') < \epsilon$. From this we can easily derive that in the Rokhlin Lemma, the base set can be taken open. For any $\delta > 0$ there thus exists a Rokhlin tower $V, T^{-1}V, \ldots, T^{-n\cdot m+1}V$ where V is an open set, and a continuous function $\tilde{\varphi}$, $0 \leq \tilde{\varphi} \leq 1$, which is zero on $X \setminus V$ such that $\lambda(\sum_{i=0}^{m\cdot n-1} \tilde{\varphi} \circ T^i = 1) > 1 - \delta$ (i.e. the sets $V, \ldots, T^{-n\cdot m+1}V$ exhaust most of X and $\tilde{\varphi}$ is close to the indicator function of V).

Define

$$W = \bigcup_{j=0}^{m-1} T^{-j\cdot n} V$$

and

$$\varphi = x_1 \sum_{j=0}^{m(1)-1} \tilde{\varphi} \circ T^{j\cdot n} + \ldots + x_r \sum_{j=m(1)+\ldots+m(r-1)}^{m(1)+\ldots+m(r)-1} \tilde{\varphi} \circ T^{j\cdot n}. \tag{1}$$

Hence, W, $T^{-1}W, \ldots, T^{-(n-1)}W$ is a Rokhlin tower, φ is zero on $X \setminus W$, and the distribution of $\sum_{i=0}^{n-1} \varphi \circ T^i$ is close to ν. For any invariant measure $\tilde{\lambda}$ we have

$$\int \varphi \, d\tilde{\lambda} = \sum_{j=1}^{r} x_j m(j) \int_V \tilde{\varphi} \, d\tilde{\lambda} = 0,$$

hence $\varphi \in \mathcal{C}_0(X)$.

Let $\epsilon > 0$ be given. As $\lim_{k\to\infty}(c_k/k) = 0$, we can choose k big enough so that $(c_k/k) \max_{1\le j\le r} |x_j| < \epsilon$. We can also suppose that n is so big that $k \cdot \lambda(W) < \epsilon$. Define $\lambda_W(.) = \lambda(. \mid W)$, the conditional probability given W.

For δ sufficiently small we get $\delta < \epsilon$ and

$$\left| \int \exp(it\varphi(x)) \, d\lambda_W(x) - \int \exp(itx) \, d\nu(x) \right| < \epsilon \quad \text{for } t \in (-k, k).$$

Define

$$f = \frac{c_k}{k} \sum_{j=0}^{n-k} \varphi \circ T^j;$$

we thus have $\|f\|_\infty < \epsilon$. For $s = k-1, \ldots, n-k$ we have

$$\int_{T^{-s}W} \exp\left(\frac{it}{c_k} S_k(f)\right) d\lambda = \int_{T^{-s}W} \exp(it \cdot \varphi \circ T^s) \, d\lambda = \lambda(W) \int \exp(it\varphi) \, d\lambda_W.$$

Therefore,

$$\left| \sum_{s=k-1}^{n-k} \int_{T^{-s}W} \exp\left(\frac{it}{c_k} S_k(f)\right) d\lambda - (n+2-2k)\lambda(W) \int \exp(itx) \, d\nu(x) \right| \le \epsilon$$

hence

$$\left| E \exp\left(\frac{it}{c_k} S_k(f)\right) - \int \exp(itx) \, d\nu(x) \right| \le 4\epsilon.$$

There thus exists a sequence of positive real numbers $(\eta_k)_k$, converging to 0, and functions $f_k \in \mathcal{C}_0(X)$ such that

$$\left| E\exp\left(\frac{it}{c_k} S_k(f_k)\right) - \int e^{itx}\, d\nu(x)\right| < \eta_k \quad \text{for } t \in [-k,k],$$
$$\lim_{k\to\infty} \|f_k\|_\infty = 0.$$

For each $k = 1,2,\dots$ there exists an open neighborhood $\mathcal{U}(f_k)$ of f_k such that for $f \in \mathcal{U}(f_k)$

$$\left| E\exp\left(\frac{it}{c_k} S_k(f)\right) - E\exp\left(\frac{it}{c_k} S_k(f_k)\right)\right| < \frac{1}{k}, \quad t \in [-k,k],$$

i.e.

$$\left| E\exp\left(\frac{it}{c_k} S_k(f)\right) - \int e^{itx}\, d\nu(x)\right| < \frac{1}{k} + \eta_k, \quad t \in [-k,k].$$

Let b_k, $k = 1,2,\dots$ be positive real numbers such that $\lim_k b_k = \infty$ and $\lim_k (b_k/c_k) = 0$. For each k define

$$\mathcal{U}_k = \{g \in \mathcal{C}_0(X) \colon \|g\|_\infty \le b_k\}$$

and

$$H_k = \bigcup_{n=k}^{\infty} [\mathcal{U}(f_n) + (U-I)\mathcal{U}_n], \quad H = \bigcap_{k=1}^{\infty} H_k.$$

The sets H_k are open and by Lemma 1, $\bigcup_{n=k}^{\infty}(U-I)\mathcal{U}_n$ is dense in $\mathcal{C}_0(X)$, so that H is a dense G_δ set. Let $f \in H$. For any positive integer k we have $f \in H_k$, thus for all $n \ge k$ there exists a decomposition $f = g' + g'' - Ug''$ with $g' \in \mathcal{U}(f_n)$, $g'' \in \mathcal{U}_n$. Hence

$$\left| E\exp\left(\frac{it}{c_n} S_n(g')\right) - \int e^{itx}\, d\nu(x)\right| \le \frac{1}{n} + \eta_n \quad \text{for } t \in [-n,n]$$

and

$$\left\| \frac{1}{c_n} S_n(g'' - Ug'')\right\|_\infty = \frac{1}{c_n} \|g'' - U^n g''\| \le \frac{2b_n}{c_n}.$$

Therefore there exists a strictly increasing sequence of integers n_k such that the distributions of $(1/c_{n_k}) S_{n_k}(f)$ weakly converge to ν. ∎

3. Rotations on the circle

3.1 ABSOLUTELY CONTINUOUS AND SMOOTH FUNCTIONS ON THE CIRCLE. Let $\mathbb{T}$ be the unit circle represented as the unit interval $[0,1)$ with the Borel σ-algebra and the Lebesgue probability measure λ. For an irrational number $\alpha \in [0,1)$, $T = T_\alpha$ denotes the rotation $x \mapsto x + \alpha \,(\mathrm{mod}\ 1)$ on $\mathbb{T}$. The functions on $\mathbb{T}$ will be often called cocycles. A cocycle F of the form $F = G - G \circ T$ is called a coboundary, G is its transfer function.

We consider the following spaces:

- $\mathcal{A}_0$ is the space of all absolutely continuous cocycles F for which $\int_0^1 F(t)\,dt = 0$,
- $\mathcal{L}_0$ is the space of all Lipschitz functions from $\mathcal{A}_0$,
- $\mathcal{C}_0^k$ is the space of all cocycles with continuous k-th derivative and with zero integral $(1 \leq k < \infty)$.

An absolutely continuous cocycle F has a derivative $f = F'$ a.e.; for $F \in \mathcal{A}_0$ we have $f \in L^1(\mathbb{T})$ and for $F \in \mathcal{L}_0 \subset \mathcal{A}_0$ we have $f \in L^\infty(\mathbb{T})$. By the definition of f, $\int_0^1 f(x)\,dx = 0$ and we have

$$F(t) = \int_0^t f(x)\,dx - \int_0^1 \int_0^u f(x)\,dxdu, \quad 0 \leq t < 1. \tag{2}$$

Therefore, $F(0) = -\int_0^1 \int_0^t f(x)\,dxdt$. Formula (2) thus defines a bijection between $\mathcal{A}_0$ and L_0^1, and between $\mathcal{L}_0$ and L_0^∞.

On the spaces $\mathcal{A}_0$ and $\mathcal{L}_0$ we introduce respectively the norms

$$\|F\|_{\mathcal{A}_0} = \int_0^1 |F'(t)|\,dt \quad \text{and} \quad \|F\|_{\mathcal{L}_0} = \operatorname*{ess\,sup}_{t\in\mathbb{T}} |F'(t)|.$$

When there is no danger of confusion the norms will be used without subscripts. $\mathcal{A}_0$ and $\mathcal{L}_0$ with the norms introduced above are Banach spaces and the relation (2) gives their isomorphisms onto L_0^1 and L_0^∞ respectively.

Let $F \in \mathcal{C}_0^k$, $1 \leq k < \infty$. Then the k-th derivative $F^{(k)}$ is a continuous function on $\mathcal{C}_0(\mathbb{T})$ and (2) shows that we can (by iterations) get F back from $F^{(k)}$. The space $\mathcal{C}_0^k$ is thus isomorphic to $\mathcal{C}_0(\mathbb{T})$ and the metric inherited from $\mathcal{C}(\mathbb{T})$ furnishes $\mathcal{C}_0^k$ with the standard topology of $\mathcal{C}_0^k$. On $\mathcal{C}_0^\infty$ we have the coarsest topology with respect to which all the functionals from all $\mathcal{C}_0^k$, $1 \leq k < \infty$, are continuous.

In the Proof of Theorem 1 we took advantage of the fact that the coboundaries are dense in $\mathcal{C}_0$. In the next two lemmas we prove that for $\mathcal{A}_0$, $\mathcal{L}_0$ and $\mathcal{C}_0^p$, $1 \leq p \leq \infty$, this holds, too.

LEMMA 2: *Let $\mathcal{E}$ be any of the spaces $\mathcal{A}_0$, $\mathcal{L}_0$ or $\mathcal{C}_0^k$, $1 \le k \le \infty$. If $F \in \mathcal{E}$ and F' is a coboundary, and $F' = G' - G' \circ T$ where $G \in \mathcal{E}$, then F is also a coboundary and $F = G - G \circ T$.*

Proof: Let $F \in \mathcal{E}$. Following (2) we have

$$F(t) = \int_0^t F'(x)\,dx - \int_0^1 \int_0^u F'(x)\,dxdu.$$

Suppose that $F' = f = g - g \circ T$ and $G(t) = \int_0^t g(x)dx$. Then

$$F(t) = \int_0^t f(x)\,dx - G(\alpha) = G(t) - G(t+\alpha). \qquad \blacksquare$$

LEMMA 3: *In the space $\mathcal{A}_0$ (resp. $\mathcal{C}_0^k$, $1 \le k \le \infty$), the set of the coboundaries*

$$G - G \circ T,$$

with G absolutely continuous (resp. $G \in \mathcal{C}_0^k$), is dense.

In $\mathcal{L}_0$, the set of coboundaries with G measurable is dense.

Proof: The set of all coboundaries $g - g \circ T$, $g \in L_0^1$, is dense in L_0^1. Their images in $\mathcal{A}_0$ in the isomorphism between L_0^1 and $\mathcal{A}_0$ are thus also dense and by Lemma 2 they are coboundaries with transfer functions from $\mathcal{A}_0$. In the same way (using Lemma 1) we prove that the coboundaries with continuously differentiable transfer functions are dense in $\mathcal{C}_0^1$. Now, Lemma 2 enables one to extend the result recursively to all $\mathcal{C}_0^p$, $1 \le p < \infty$.

The set of coboundaries $F = G - G \circ T$ with $G \in \mathcal{C}^\infty$ is dense in every $\mathcal{C}_0^p$, $1 \le p < \infty$, hence it is also dense in $\mathcal{C}_0^\infty$.

The set of coboundaries with integrable transfer function is not dense in L_0^∞ (see [8] or [9]) and similarly, we can easily prove that the coboundaries with absolutely continuous transfer functions are not dense in $\mathcal{L}_0$. The proof of the Lemma for $\mathcal{L}_0$ is given in [21] (and is much more complicated than in the previous cases). $\blacksquare$

3.2 THE DENJOY–KOKSMA THEOREM. Let $\xi = (x_n)_{n \ge 0}$ be a sequence in $\mathbb{T}$. The discrepancy to the origin $D_N^*(\xi)$ of ξ is by definition the quantity

$$D_N^*(\xi) = \max_{0 \le t < 1} \left| t - \frac{1}{N} \sum_{0 \le n < N} \mathbf{1}_{[0,t)}(x_n) \right|.$$

For $x_n = T^n x = n\alpha + x \bmod 1$, $x \in [0,1) = \mathbb{T}$, we have $D_N^*(\xi) = o(1)$, i.e. ξ is uniformly distributed mod 1.

For all maps $f\colon \mathbb{T} \to \mathbb{R}$ of bounded variation $V(f)$ one has the so-called Denjoy–Koksma inequality

$$(3) \qquad \Big| N \int_0^1 f(t)\,dt - \sum_{0 \le n < N} f(x_n) \Big| \le V(f) N D_N^*(\xi)$$

(see e.g. [13], Theorem 5.1, p.143). If α has bounded partial quotients, there exists a constant $c = c(\alpha)$ for which $ND_N^* \le 3 + c \log N$ (see [13, Theorem 3.4] where c is given); if α is of type η (see the next paragraph 3.3 for the definition), then by [13, Theorem 3.2] for every $\epsilon > 0$, $ND_N^* = O(N^{(-1/\eta)+\epsilon})$. Therefore, the rate of convergence in the Ergodic Theorem cannot be arbitrarily slow already for the functions with bounded variation.

3.3 Diophantine approximation and discrepancy. In the sequel we shall need results on the discrepancy of sequences $(n\alpha \pmod 1)_n$. To this aim we recall some basic facts on continued fraction expansion. By $\|x\|$, $x \in [0,1)$, we shall denote $\min\{x, 1-x\}$. For $x \in \mathbb{R}$ let $[x]$ denote the integer part of x and $\{x\} = x - [x]$ the fractional part.

Let $\alpha = [0; a_1, a_2, \ldots]$ be the continued fraction expansion of $\alpha \in [0,1)$. The convergents p_n/q_n of x are given by the following recurrent formulas (see [11])

$$(4) \qquad p_{n+1} = a_{n+1} p_n + p_{n-1} \quad \text{and} \quad q_{n+1} = a_{n+1} q_n + q_{n-1}$$

for $n \ge 1$, with $p_0 = 0$, $p_1 = 1$, $q_0 = 1$, $q_1 = a_1$. We recall the following basic formulas (see [11]):

$$(5) \qquad \frac{1}{2q_{n+1}} \le \|q_n\alpha\| \le \frac{1}{q_{n+1}}, \qquad \|q_n\alpha\| = |q_n\alpha - p_n| = \min_{1 \le |m| < q_{n+1}} \|m\alpha\|,$$

$$(6) \qquad \|q_{n-2}\alpha\| = a_n \|q_{n-1}\alpha\| + \|q_n\alpha\|,$$

$$(7) \qquad q_{n+1}\|q_n\alpha\| + q_n\|q_{n+1}\alpha\| = 1.$$

We thus have in particular

$$(8) \qquad a_n \|q_{n-1}\alpha\| \ge \frac{1}{4q_{n-1}}$$

and

$$(9) \qquad q_{n-1} a_n \|q_{n-1}\alpha\| \ge 1 - \frac{2}{a_n}.$$

Indeed, for $a_n = 1$ we have $q_n = q_{n-1} + q_{n-2} \le 2q_{n-1}$, hence $a_n\|q_{n-1}\alpha\| \ge 1/2q_n \ge 1/4q_{n-1}$. For $a_n \ge 2$ we have

$$a_n\|q_{n-1}\alpha\| \ge a_n/2q_n \ge a_n/2(a_nq_{n-1} + q_{n-2}) \ge 1/4q_{n-1}.$$

This proves (8). The inequality (9) is also a consequence of (5) and (7). In fact

$$\begin{aligned} q_{n-1}a_n\|q_{n-1}\alpha\| &= q_n\|q_{n-1}\alpha\| - q_{n-2}\|q_{n-1}\alpha\| \\ &= 1 - q_{n-1}\|q_n\alpha\| - q_{n-2}\|q_{n-1}\alpha\| \\ &\ge 1 - (q_{n-1}/q_{n+1}) - (q_{n-2}/q_n) \\ &\ge 1 - (2q_{n-1}/q_n) \ge 1 - (2/a_n) \end{aligned}$$

as required.

We say that α is of type η if

$$\eta = \inf\{\tau \in \mathbb{R}\colon \text{there exists } c > 0 \text{ such that for all } q \in \mathbb{N},\ q^\tau\|q\alpha\| \ge c\}$$

(see [13], Lemma 3.1, p. 121); if for every $c > 0$ and $\tau > 0$ there exists $q \in \mathbb{N}$ with $q^\tau\|q\alpha\| < c$, we say that α is of type infinity. The next statement gives some useful characterizations of the type.

LEMMA 4: *For any irrational number $\alpha \in [0,1)$, the following are equivalent:*

(i) *α is of type η,*

(ii) $\eta = \inf\{\tau \in \mathbb{R}\colon \exists\, c > 0,\ \forall\, n \ge 0,\ a_{n+1} \le q_n^{\tau-1}/c\}$,

(iii) $\eta = \inf\{\tau \in \mathbb{R}\colon \exists\, c > 0,\ \forall\, n \ge 0,\ q_{n+1} \le q_n^{\tau}/c\}$.

Proof: Let $\tau > \eta$ and let $c > 0$ be such that $q^\tau\|q\alpha\| \ge c$ for all natural numbers q. Then for all indices n one has (cf. (4), (5))

$$c \le q_n^\tau\|q_n\alpha\| \le q_n^\tau\frac{1}{q_{n+1}} \le \frac{q_n^{\tau-1}}{a_{n+1}}.$$

Therefore, $a_{n+1} \le q_n^{\tau-1}/c$. Reciprocally, assume that for $\tau \ge 1$ there exists $c > 0$ such that $a_{n+1} \le c \cdot q_n^{\tau-1}$ for all $n \ge 0$. Then, for all $q \in \mathbb{N}$ and n such that $q_n \le q < q_{n+1}$ we get successively (cf. (5))

$$q^\tau\|q\alpha\| \ge q_n^\tau\|q_n\alpha\| \ge q_n^\tau\frac{1}{2(a_{n+1}q_n + q_{n-1})} \ge \frac{1}{4}\frac{q_n^{\tau-1}}{a_{n+1}} \ge \frac{1}{4c}.$$

Therefore (i) is equivalent to (ii). The proof of the equivalence between (ii) and (iii) is left to the reader. ∎

3.4 ROKHLIN TOWERS FOR ROTATIONS. Let us suppose that n is odd so that $q_{n-1}\alpha - p_{n-1}$ is positive. Then $[0,1)$ splits into two Rokhlin towers:

$$[\{j\alpha\}, \{(q_{n-1}+j)\alpha\}), \quad j = 0, \ldots, q_n - 1$$

is the bigger one and

$$[\{q_n\alpha\}, 1), \ [\{(j+q_n)\alpha\}, \{j\alpha\}), \quad j = 1, \ldots, q_{n-1} - 1$$

is the smaller one.

For n even we get

$$[\{q_{n-1}\alpha\}, 1), \ [\{(q_{n-1}+j)\alpha\}, \{j\alpha\}), \quad j = 1, \ldots, q_n - 1$$

as the bigger tower and

$$[\{j\alpha\}, \{(j+q_n)\alpha\}), \quad j = 0, \ldots, q_{n-1} - 1$$

as the smaller one.

4. Rotations with unbounded partial quotients

In this section α will denote an irrational number in $[0,1)$ with unbounded partial quotients. We shall study the weak convergence of distributions of sums $\frac{1}{c_k} S_{B_k}(F)$ where $B_k \to \infty$, $c_k \to \infty$ or $c_k \equiv 1$ for all k, F is from one of the spaces $\mathcal{A}_0$, $\mathcal{L}_0$, $\mathcal{C}_0^p$, $1 \leq p \leq \infty$.

The result, Theorem 2, immediately gives as a corollary a rate of the growth of the partial sums $S_n(F)$. The next two theorems show that the rate is in some sense the best possible. As a consequence of all three theorems we get known (cf. e.g. [1] or [8]) necessary and sufficient conditions of the existence and genericity of ergodic cocycles in $\mathcal{C}_0^p$, $1 \leq p \leq \infty$.

Let p be a positive integer. If $\limsup_{n\to\infty} q_n/q_{n-1}^p = \infty$, there exist positive integers n_k, B_k, c_k, for which $\lim_k n_k = \infty$ and

$$B_k/q_{n_k} \to 0, \quad c_k q_{n_k-1}^p / B_k \to 0, \quad \text{and} \quad c_k \to \infty,$$

(10) or

$$B_k/q_{n_k} \to 0, \quad c_k q_{n_k-1}^p / B_k \to 0, \quad c_k = 1 \quad \text{for all } k \quad \text{and} \quad \{B_k\alpha\} \to 0.$$

THEOREM 2: *Let $\mathcal{E}$ be one of the spaces $\mathcal{A}_0$, $\mathcal{L}_0$, $\mathcal{C}_0^p$, $1 \leq p \leq \infty$. If*

(a) (10) *holds true for $p = 1$ and $\mathcal{E}$ equals $\mathcal{A}_0$ or $\mathcal{L}_0$ or $\mathcal{C}_0$, or*

(b) (10) *holds true for $p = r$ and $\mathcal{E} = \mathcal{C}_0^r$, $1 \leq r < \infty$, or*

(c) (10) *holds true for all positive integers p and $\mathcal{E} = \mathcal{C}_0^\infty$,*

then there exists a dense G_δ set of $F \in \mathcal{E}$ such that

(11) *the distributions of $\frac{1}{c_k} S_{B_k}(F)$ are a dense set in the space of all probability measures on $\mathbb{R}$.*

Remark: The sequences $(n_k)_k$, $(B_k)_k$, $(c_k)_k$, for which the assumptions of Theorem 2 are fulfilled, can be found:

for $\mathcal{E} = \mathcal{A}_0, \mathcal{L}_0$ if α has unbounded partial quotients,

for $\mathcal{E} = \mathcal{C}_0^p$ if the type of α is greater than p,

for $\mathcal{E} = \mathcal{C}_0^\infty$ if α is of type infinity.

Taking in (10) the sequence of c_k constant has been motivated by the following application: For a real cocycle F on the circle we define the skew product T_F, i.e. the transformation

$$T_F(x, y) = (Tx, y + F(x))$$

on the cylinder $\mathbb{T} \times \mathbb{R}$ preserving the product measure. If the transformation T_F is ergodic, we usually say that the cocycle is ergodic.

If α is as in the Remark, the assumptions of Theorem 2 are fulfilled with $c_k = 1$ for all k, hence for a dense G_δ subset of $F \in \mathcal{E}$ the subsequences of $S_{B_k}(F)$ converge in probability to all constants. Hence, the Essential Value Condition which is sufficient for ergodicity of F (see [19]) is fulfilled, namely:

For every set B of positive measure, $a \in \mathbb{R}$ and $\epsilon > 0$, there exists n such that

$$\lambda(B \cap T^{-n}B \cap \{a - \epsilon < S_n(F) < a + \epsilon\}) > 0.$$

As a corollary to Theorem 2 we thus get

THEOREM 3: *Under the same assumptions as in Theorem 2, there exists a dense G_δ set of ergodic cocycles $F \in \mathcal{E}$.*

The result has been known for smooth functions, for absolutely continuous ones it seems to be new. Theorem 3 (as well as the corresponding version of

Theorem 2 using $c_k \equiv 1$) was found during the second author's stay in Toruń thanks to Professor Mariusz Lemańczyk.

Proof of Theorem 2: Let us first suppose that $\mathcal{E} = \mathcal{C}_0^p$, $1 \le p < \infty$. Similarly as in the proof of Theorem 1, Γ denotes a countable and dense set of probability measures ν on $\mathbb{R}$ supported by finite sets of (now, not necessarily rational) numbers and $\int x\, d\nu(x) = 0$.

We suppose that $\nu \in \Gamma$ is fixed; similarly as in the proof of Theorem 1 it suffices to show that (11) holds for a dense G_δ set of $F \in \mathcal{E}$. From now on, functions on the circle will be understood as 1-periodic functions on the real line.

Let H be a step function on $[0,1)$ with $\lambda \circ H^{-1} = \nu$; H is constant on the intervals (x_i, x_{i+1}), $0 = x_0 < x_1 < \cdots < x_m = 1$. Let

$$0 < \delta < \min_{0 \le i \le m-1} (x_{i+1} - x_i)/3,$$

$x_i' = x_i + \delta$, $x_i'' = x_{i+1} - \delta$, $0 \le i \le m-1$.

We shall define a function $H_1 \in \mathcal{E}$ which equals H on the intervals (x_i', x_i''), $|H_1| \le |H|$ and $H_1^{(j)}(0) = 0 = H_1^{(j)}(1)$, $0 \le j \le p$ ($H_1^{(0)} = H_1$, p is from the definition of $\mathcal{E}$). In fact, there exists a function $h \in \mathcal{C}^\infty$ which is of constant sign on the intervals (x_i, x_i'), and on (x_i'', x_{i+1}) (the sign of 0 is + by definition), equals zero on (x_i', x_i''),

$$\int_{x_i}^{x_i'} h\, d\lambda = -\int_{x_i''}^{x_{i+1}} h\, d\lambda = H\left(\frac{x_i + x_{i+1}}{2}\right),$$

for $i = 0, \dots, m-1$ and finally

$$h^{(k)}(0) = h^{(k)}(1)$$

for all $k \ge 0$.

For any positive integer p, the function

$$H_1(t) = \int_0^t h(z)\, dz$$

fulfills our needs. Moreover, for a given p put $h_1 = h^{(p-1)}$ for short; then we still have

$$H_1(t) = \int_0^t \int_0^{x_{p-1}} \cdots \int_0^{x_1} h_1(x)\, dx dx_1 \dots dx_{p-1}.$$

Choosing δ sufficiently small, we can have $H_1 = H$ on $[0,1)$ up to a set of arbitrarily small (positive) measure.

The interval $[0,1)$ splits into two Rokhlin towers as shown in Section 3.4. For simplicity (and without loss of generality) we suppose that $n = n_k$ is odd and we define new Rokhlin towers $J_0, \dots, J_{q_{n-1}-1}$ by

$$J_0 = [0, a_n\|q_{n-1}\alpha\|)\Big(= \bigcup_{j=0}^{a_n-1} T^{jq_{n-1}}[0, \|q_{n-1}\alpha\|)\Big),$$

$$J_i = T^i J_0 \quad \text{for } i = 1, \dots, q_{n-1}-1.$$

The intervals J_i are mutually disjoint and by (9),

$$\lambda\Big(\bigcup_{i=0}^{q_{n-1}-1} J_i\Big) = q_{n-1}a_n\|q_{n-1}\alpha\| \geq 1 - \frac{2}{a_n}.$$

Let us denote $n = n_k$ when this causes no confusion. Define

$$h_2(x) = \frac{1}{(a_n\|q_{n-1}\alpha\|)^p} h_1\left(\frac{x}{a_n\|q_{n-1}\alpha\|}\right), \quad x \in J_0,$$

$$f(x) = \begin{cases} \frac{c_k}{B_k} h_2(T^{-i}x) & \text{for } x \in J_i, \quad i = 0, \dots, q_{n-1}-1 \\ 0 & \text{for } \quad x \in [0,1) \smallsetminus \bigcup_{i=0}^{q_{n-1}-1} J_i, \end{cases}$$

$$F(t) = \int_0^t \int_0^{x_{p-1}} \cdots \int_0^{x_1} f(x)\,dx dx_1 \dots dx_{p-1}, \quad t \in [0,1).$$

We thus have

$$F^{(p)} = f,$$

$$\|f\|_\infty = \frac{c_k}{B_k} \frac{\|h_1\|_\infty}{(a_n\|q_{n-1}\alpha\|)^p} \leq \frac{c_k}{B_k} 2^p q_{n-1}^p \|h_1\|_\infty$$

(as by (5) and (4), $a_n\|q_{n-1}\alpha\| \geq a_n/2q_n \geq 1/2q_{n-1}$). By the assumption (10) we have $\lim_k c_k q_{n_k-1}^p / B_k = 0$, hence for every $\epsilon > 0$ we can find k big enough so that $\|F\|_{C,p} = \|f\|_\infty < \epsilon$. A straightforward computation gives

$$\frac{B_k}{c_k} F(x) = H_1\left(\frac{x}{a_n\|q_{n-1}\alpha\|}\right) \quad \text{on } J_0,$$

$$\frac{B_k}{c_k} F(x) = \frac{B_k}{c_k} F(T^{-i}x) \quad \text{on } J_i, \ i = 1, \dots, q_{n-1}-1, \ \text{and}$$

$$\frac{B_k}{c_k} F(x) = 0 \quad \text{on } [0,1) \smallsetminus \bigcup_{i=0}^{q_n-1} J_i.$$

Denote $b_k = [B_k/q_{n-1}]$. By the definition, the functions $f \circ T^\ell$ where $\ell = -i - jq_{n-1}$, $0 \le i \le q_{n-1} - 1$, $0 \le j \le b_k - 1$, have the same values on the intervals

$T^r(a_n\|q_{n-1}\alpha\|x'_s, a_n\|q_{n-1}\alpha\|x''_s)$, $0 \le r \le q_{n-1} - 1$, $0 \le s \le m - 1$, except on intervals at both extremities of lengths at most $b_k\|q_{n-1}\alpha\| = (b_k/a_n)\lambda(J_0)$; moreover $\lim_k(b_k/a_{n_k}) = 0$. Therefore, for any $\epsilon > 0$ we can choose δ sufficiently small and k sufficiently large such that $B_kF = S_{B_k}(F)$ with probability bigger than $1 - \epsilon$.

For each of the intervals J_j, $j = 0, \ldots, q_{n-1} - 1$,

$$\int_{J_j} e^{itB_kF(x)/c_k}dx = \int_0^{a_n\|q_{n-1}\alpha\|} e^{itB_kF(x)/c_k}dx = a_n\|q_{n-1}\alpha\| \int_0^1 e^{itH_1(x)}dx;$$

from this and from $1 \ge q_{n-1}a_n\|q_{n-1}\alpha\| \ge 1 - (2/a_n)$ (cf. (5), (4), (9)) we get

$$\begin{aligned}
&\left|\int_0^1 e^{itB_kF(x)/c_k}dx - \int_0^1 e^{itH_1(x)}dx\right| \\
=&\left|\int_{[0,1)\setminus\bigcup_{j=0}^{q_{n-1}-1} J_j} e^{itB_kF(x)/c_k}\,dx - (1 - q_{n-1}a_n\|q_{n-1}\alpha\|)\int_0^1 e^{itH_1(x)}\,dx\right| \\
\le&\frac{4}{a_n}.
\end{aligned}$$

For any $\epsilon > 0$, we can thus find $F \in \mathcal{E}$ and a positive integer $k(\epsilon)$ such that $\|F\|_{C,p} < \epsilon$ and

$$\left|\int_0^1 \exp(itH_1)\,d\lambda - \int_0^1 \exp(itS_{B_k}(F)/c_k)\,d\lambda\right| < \epsilon$$

for all $t \in \mathbb{R}$ and $k \ge k(\epsilon)$. The function H_1 can be found equal to H on $[0,1)$ up to a set of arbitrarily small (positive) measure, hence there exist F and a new $k(\epsilon)$ such that

$$\left|\int_0^1 \exp(itH)\,d\lambda - \int_0^1 \exp(itS_{B_k}(F)/c_k)\,d\lambda\right| < \epsilon \quad \text{for } t \in \mathbb{R} \text{ and } k \ge k(\epsilon).$$

There thus exists a sequence $F_k \in \mathcal{C}_0^p$ such that the norms of F_k in C^p converge to zero and the distributions of $(1/c_k)S_{B_k}(F_k)$ weakly converge to ν. From this and the density of the set of coboundaries in $\mathcal{C}_0^p$ we can derive (11) using the same arguments as in the proof of Theorem 1.

The proofs for $\mathcal{E} = \mathcal{A}_0$ and $\mathcal{E} = \mathcal{L}_0$ follow from $\|F_k\|_{\mathcal{A}} \le \|F_k\|_{\mathcal{L}} \le \|F_k\|_{C^1}$.

It remains to prove the Theorem for $\mathcal{E} = \mathcal{C}_0^\infty$.

By the previous construction there exists a sequence of $F_k \in \mathcal{C}_0^\infty$, $k \in K$, where K is an infinite set of positive integers, the distributions of $(1/c_k)S_{B_k}(F_k)$ weakly converge to ν, and the sum $\sum_{k\in K} \|F_k\|_{C^p}$ converges for every $1 \le p < \infty$. By Lemma 3 we can replace each F_k by a coboundary $G_k - G_k \circ T$ with $G_k \in \mathcal{C}^\infty$. Because $\|B_k\alpha\| \to 0$ as $k \to \infty$ (recall that $\|x\|$ denotes the distance of x from the set of integers), we can (replacing K by a suitable infinite subset) guarantee that $(1/c_k)\sum_{j<k, j\in K} S_{B_k}(F_j) \to 0$ in the measure as $K \ni k \to \infty$. From the decay of the norms of F_k it follows that for a suitable infinite subset of K we also have $(1/c_k)\sum_{j>k,j\in K}^\infty S_{B_k}(F_j) \to 0$ in the measure. Hence, for $F = \sum_{k\in K} F_k \in \mathcal{C}_0^\infty$ the distributions of $(1/c_k)S_{B_k}(F)$ weakly converge to ν. Because the coboundaries are dense in $\mathcal{C}_0^\infty$, we can derive (11) using the arguments from the proof of Theorem 1 once again. ∎

THEOREM 4: *Let $F \in \mathcal{C}_0^p$. Then there exist positive numbers $C, B_1, B_2, \dots$ such that*

$$\sum_{i=1}^\infty B_i^2 \le \int (F^{(p)})^2(x)\,dx$$

and for any positive integer q,

$$|S_q(F)| \le C\left(B_n \frac{q}{q_{n-1}^p} + B_{n-1}\frac{q_{n-1}}{q_{n-2}^p} + \cdots + B_1 \frac{q_1}{q_0^p}\right) \tag{12}$$

and

(13)
$$\int S_q^2(F)(x)\,dx \le C^2\left(B_n^2\left(\frac{q}{q_{n-1}^p}\right)^2 + B_{n-1}^2\left(\frac{q_{n-1}}{q_{n-2}^p}\right)^2 + \cdots + B_1^2\left(\frac{q_1}{q_0^p}\right)^2\right),$$

where $q_{n-1} \le q \le q_n$.

As a corollary to Theorems 2 and 4 we get

THEOREM 5: *Let α be of type η, $p \ge 1$ be an integer and F be an arbitrary function from $\mathcal{C}_0^p$.*

(1) *If $\eta < p$, then there exists a constant C such that*

$$|S_n(F)| < C$$

for all n.

(2) *If $\eta > p$, then for every $\epsilon > 0$ there exists an integer $\bar{q}$ such that for every $q \geq \bar{q}$,*

$$|S_q(F)| \leq q^{1-\frac{p}{\eta}+\epsilon} \tag{14}$$

and

(3) *there exists a dense G_δ set of $F \in \mathcal{C}_0^p$ such that*

$$\lambda(|S_q(F)| > q^{1-\frac{p}{\eta}-\epsilon}) > 1-\epsilon \tag{15}$$

for infinitely many positive integers q.

Remarks: If we take $p = 1$, Theorem 5 remains valid and the proof works for $\mathcal{A}_0$ and $\mathcal{L}_0$ as well as for $\mathcal{C}_0^1$. The upper bound in this case can be also found in [13] by combining Theorem 3.2, p. 123, which gives an estimate of the discrepancy of $\xi = (\{i\alpha\})_{i\geq 0}$ where $\alpha \in (0,1)$ is an irrational number of type η, $D_N^*(\xi) = O(N^{(-1/\eta)+\epsilon})$, and the theorem of Koksma–Denjoy which holds for all functions of bounded variation. Theorem 5 thus extends the result from [13] to k-times continuously differentiable functions and shows that the bound can be approached for a generic set of functions.

By Theorem 5, for $\eta < p$ the partial sums $S_q(F)$ are bounded, hence F is a coboundary. On the other hand, if $\eta > p$, there exists a dense G_δ set of $F \in \mathcal{C}_0^p$ for which the partial sums $S_q(F)$ are not stochastically bounded, hence F are not coboundaries (according to [15]).

Using Theorem 2 and Theorem 4 more directly, we can get that

- if $\limsup_{n\to\infty} q_n/q_{n-1}^p = \infty$, there exists a dense G_δ set of $F \in \mathcal{C}_0^p$ which are not coboundaries;
- if $\limsup_{n\to\infty} q_n/q_{n-1}^p < \infty$, the L^2 norms of the sums $S_q(F)$ are for every $F \in \mathcal{C}_0^p$ bounded, hence F is a coboundary with a transfer function in L^2 (see, e.g. [17]).

This reproves the well known result saying that if the limes inferior of $q^p\|q\alpha\|$ is zero, all functions from $\mathcal{C}_0^p$ are coboundaries, while in the other case there exists a G_δ set of $F \in \mathcal{C}_0^p$ which are not coboundaries (cf., e.g. [1]). From Theorem 3 it follows that instead of "are not coboundaries" we can say "are ergodic". Using methods from the proof of Theorem 7, a function $F \in \mathcal{C}_0^p$ can be found, for which $|S_q(F)|$ are not bounded while $\limsup_{n\to\infty} q_n/q_{n-1}^p$ is positive and bounded. By Theorem 4, F is a coboundary with an unbounded transfer function.

Proof of Theorem 4: Let

$$F(x) = \sum_{k\in\mathbb{Z}} b_k e^{2\pi ikx}$$

be the Fourier expansion of F. From $\int_0^1 F(x)\,dx = 0$ follows $b_0 = 0$. The function F is real, hence $\overline{b_k} = b_{-k}$ for all k. Let

$$b'_k = |k|^p b_k, \quad k \in \mathbb{Z}.$$

We thus have

$$F(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{b'_k}{|k|^p} e^{2\pi ikx};$$

as $F^{(p)} \in L_0^2$, $\sum_{k\in\mathbb{Z}} |b'_k|^2 < \infty$. Let

$$F_1(x) = \sum_{|k|\le q_{n-1}-1} b_k e^{2\pi ikx} \quad \text{and} \quad F_2(x) = \sum_{|k|\ge q_{n-1}} b_k e^{2\pi ikx}.$$

For any integer $q \ge 2$ we have

$$|S_q(F_1)| \le \sum_{1\le|k|\le q_{n-1}-1} |b_k|\cdot|S_q(e^{2\pi ik\alpha})|, \quad |S_q(F_2)| \le \sum_{|k|\ge q_{n-1}} |b_k|\cdot|S_q(e^{2\pi ik\alpha})|.$$

Let $q_{n-1} \le q < q_n$. Classically

$$S_q(e^{2\pi ik\alpha}) = \frac{1-e^{2\pi ikq\alpha}}{1-e^{2\pi ik\alpha}}$$

and following [13], pp. 122–123, we have

$$\frac{1}{|e^{2\pi ik\alpha}-1|} \le \frac{1}{2\|k\alpha\|},$$

hence

$$|S_q(e^{2\pi ik\alpha})| \le \frac{1}{\|k\alpha\|}. \tag{16}$$

Therefore,

$$|S_q(F_1)| \le \sum_{j=1}^{n-1} \sum_{|k|=q_{j-1}}^{q_j-1} \frac{|b'_k|}{|k|^p}\cdot|S_q(e^{2\pi ik\alpha})|$$

$$\le \sum_{j=1}^{n-1} \frac{1}{q_{j-1}^p} \sum_{|k|=q_{j-1}}^{q_j-1} \frac{|b'_k|}{\|k\alpha\|}.$$

From the structure of the Rokhlin towers for rotations (see 3.4) and (5) it follows that

$$\sum_{k=1}^{q_m-1} \frac{1}{\|k\alpha\|^2} \le \sum_{k=1}^{q_m-1} \frac{1}{\{k\alpha\}^2} + \sum_{k=1}^{q_m-1} \frac{1}{\{-k\alpha\}^2}$$
$$\le \frac{2}{\|q_{m-1}\alpha\|^2} \cdot \sum_{k=1}^{q_m-1} \frac{1}{k^2} \le 8q_m^2 \sum_{k=1}^{q_m-1} \frac{1}{k^2}, \qquad m = 1, 2, \ldots.$$

($\{x\}$ denotes the fractional part of x.)

Let $K^2 = \sum_{j=1}^{\infty} 1/j^2$; then by the Schwartz inequality,

$$\sum_{|k|=q_{j-1}}^{q_j-1} \frac{|b'_k|}{\|k\alpha\|} \le 2\sqrt{2} K q_j B_j \le 3K q_j B_j$$

where $B_j^2 = \sum_{|k|=q_{j-1}}^{q_j-1} |b'_k|^2$, $j = 1, 2, \ldots$, hence

$$|S_q(F_1)| \le 3K \sum_{j=1}^{n-1} B_j \cdot \frac{q_j}{q_{j-1}^p}. \tag{17}$$

The functions $x \mapsto S_q(e^{2\pi i k x})$ are mutually orthogonal, hence for

$$F_{1,j}(x) = \sum_{|k|=q_{j-1}}^{q_j-1} \frac{b'_k}{|k|^p} \cdot S_q(e^{2\pi i k x}),$$

$j = 1, \ldots, n-1$, we have

$$E\ F_1^2 = E \sum_{j=1}^{n-1} |F_{1,j}|^2 \le 9K^2 \sum_{j=1}^{n-1} B_j^2 \cdot \left(\frac{q_j}{q_{j-1}^p}\right)^2. \tag{18}$$

Now, we shall give an estimate for F_2. From the structure of the Rokhlin tower described in 3.4 it follows that for any interval J of length $\|q_{n-2}\alpha\|$, all the intervals $T^j(J)$, $j = 0, \ldots, q_{n-1} - 1$ are disjoint and the same result is true if we replace α by $-\alpha$. Assume that n is even (the remaining case is analogous), $n \ge 2$, and let K_n be the interval (mod 1) $[-\frac{1}{2}\|q_{n-2}\alpha\|, \frac{1}{2}\|q_{n-2}\alpha\|)$. For each $x \in K_n$ let $\ell(x)$ be the smallest integer $\ell > 0$ such that $T^\ell(x) \in K_n$. The function $\ell(\cdot)$ takes only two values. In fact $\ell = q_{n-1}$ on the interval $[-\frac{1}{2}\|q_{n-2}\alpha\| + \|q_{n-1}\alpha\|, \frac{1}{2}\|q_{n-2}\alpha\|)$ and $\ell = q_{n-1} + q_{n-2}$ otherwise. Let $(r_j)_j$ be the increasing sequence of all natural numbers $r \geq q_{n-1}$ such that $r\alpha \in K_n$. Hence, for each $j = 1, 2, \dots$ we have $r_{j+1} - r_j \in \{q_{n-1}, q_{n-1} + q_{n-2}\}$ and for all $x \in K_n$ we still get from the Rokhlin tower and (5)

$$
\begin{aligned}
\sum_{j=1}^{\ell(x)-1} \frac{1}{\|x + j\alpha\|^2} &\leq \sum_{j=1}^{\ell(x)-1} \left(\frac{1}{\{x + j\alpha\}^2} + \frac{1}{\{-x - j\alpha\}^2} \right) \\
&\leq \frac{2}{\frac{1}{4}\|q_{n-2}\alpha\|^2} \sum_{j=1}^{\ell(x)-1} \frac{1}{j^2} \\
&\leq 8K^2 \frac{1}{\|q_{n-2}\alpha\|^2} \leq 32K^2 q_{n-1}^2.
\end{aligned} \tag{19}
$$

It follows from (16) and (19) that

$$\sum_{|k|=r_j+1}^{r_{j+1}-1} \frac{|b'_k|}{|k|^p} |S_q(e^{2\pi i k\alpha})| \leq \sum_{|k|=r_j+1}^{r_{j+1}-1} \frac{|b'_k|}{|k|^p \|k\alpha\|} \leq 16K \cdot q_{n-1} \cdot B''_j / r_j^p$$

where $B''_j = (\sum_{|k|=r_j+1}^{r_{j+1}-1} |b'_k|^2)^{1/2}$, $j = 1, 2, \dots$. From $r_j \geq j \cdot q_{n-1}$ we get

$$\sum_{j=1}^{\infty} \sum_{|k|=r_j+1}^{r_{j+1}-1} \frac{|b'_k|}{|k|^p} |S_q(e^{2\pi i k\alpha})| \leq 16K \frac{q_{n-1}}{q_{n-1}^p} \sum_{j=1}^{\infty} \frac{B''_j}{j^p} \leq 16B'' K^2 \frac{q_{n-1}}{q_{n-1}^p}$$

where $B'' = (\sum_{j=1}^{\infty} B''^2_j)^{1/2}$. Finally

$$\sum_{j=1}^{\infty} \frac{|b'_{r_j}|}{r_j^p} |S_q(e^{2\pi i r_j \alpha})| \leq q \sum_{j=1}^{\infty} \frac{|b'_{r_j}|}{(j \cdot q_{n-1})^p} \leq B'_{n-1} K \frac{q}{q_{n-1}^p}$$

where $B'_{n-1} = (\sum_{j=1}^{\infty} |b'_{r_j}|^2)^{1/2}$. Therefore,

$$|S_q(F_2)| \leq 2B'_{n-1} K \cdot \frac{q}{q_{n-1}^p} + 16K^2 B'' \frac{q_{n-1}}{q_{n-1}^p};.$$

from this and from (17) we get (12).

Similarly as in the case of F_1, we can show that

$$EF_2^2 \leq C_1^2 \left(\left(\frac{q}{q_{n-1}^p} \right)^2 + \left(\frac{q_{n-1}}{q_{n-1}^p} \right)^2 \right),$$

where C_1 is a constant. From this and from (18) we get (13). ∎

Proof of Theorem 5: 1. Let $\epsilon, \epsilon' > 0$, $1-(p/\eta)+\epsilon < 0$, and $\frac{p}{\eta} - \frac{p}{\eta+\epsilon'} < \epsilon$. By Lemma 4(iii) there exists $c \geq 1$ such that $q_k \leq c \cdot q_{k-1}^{\eta+\epsilon'}$ for every $k \geq 1$. We thus have

$$q_k^{\frac{p}{\eta+\epsilon'}} \leq c^{\frac{p}{\eta+\epsilon'}} q_{k-1}^p,$$

hence

$$\frac{q_k}{q_{k-1}^p} \leq c^{\frac{p}{\eta+\epsilon'}} q_k^{1-\frac{p}{\eta+\epsilon'}} \leq c^{\frac{p}{\eta}} q_k^{1-\frac{p}{\eta}+\epsilon}.$$

Let n, q be positive integers, $q_{n-1} \leq q < q_n$. We have $q \leq c \cdot q_{n-1}^{\eta+\epsilon'}$, therefore as above

$$\frac{q}{q_{n-1}^p} \leq c^{\frac{p}{\eta}} \cdot q^{1-\frac{p}{\eta}+\epsilon}.$$

The numbers B_i are bounded, hence for some constant D (not depending on q),

(20)

$$|S_q(F)| \leq D\Big(\frac{q}{q_{n-1}^p} + \frac{q_{n-1}}{q_{n-2}^p} + \cdots + \frac{q_1}{q_0^p}\Big) \leq D \cdot c^{\frac{p}{\eta}} \big(q^{1-\frac{p}{\eta}+\epsilon} + q_{n-1}^{1-\frac{p}{\eta}+\epsilon} + \cdots + q_1^{1+\frac{p}{\eta}+\epsilon}\big).$$

From (4) we get $q_k \geq f_k$, $2q_{k-1} \leq q_{k+1}$, $k = 1, 2, \ldots$ where $(f_k)_k$ is the Fibonacci sequence ($f_0 = f_1 = 1$ and $f_{k+1} = f_k + f_{k-1}$); if $1 - p/\eta + \epsilon < 0$, then

$$|S_q(F)| < 2Dc^{\frac{p}{\eta}} \sum_{k=1}^{\infty} f_k^{1-\frac{p}{\eta}+\epsilon} < \infty.$$

2. If $1 - p/\eta > 0$ and $\epsilon > 0$, then from $2q_{n-1} \leq q_{n+1}$ (see (4)) it follows that for some constant E,

$$q^{1-\frac{p}{\eta}+\epsilon} + q_{n-1}^{1-\frac{p}{\eta}+\epsilon} + \cdots + q_1^{1-\frac{p}{\eta}+\epsilon} \leq E \cdot q^{1-\frac{p}{\eta}+\epsilon}.$$

From this and from (20) it follows that there exists a constant K,

$$|S_q(F)| \leq K(1 + q^{1-\frac{p}{\eta}+\epsilon}) \quad \text{for all } q \in \mathbb{N}.$$

The second statement of Theorem 5 easily follows.

3. By similar arguments as above we can derive that for any $\epsilon > 0$ there exist infinitely many n for which

$$q_n^{1-\frac{p}{\eta}-\epsilon} < \frac{q_n}{q_{n-1}^p}. \tag{21}$$

In fact, choose $0 < \epsilon' < \eta$ such that $\frac{p}{\eta-\epsilon'} - \frac{p}{\eta} < \epsilon$. There are infinitely many n with $q_{n-1}^{\eta-\epsilon'} < q_n$ so that $q_n^{1-\frac{p}{\eta-\epsilon'}} < q_n/q_{n-1}^p$ and (21) follows.

We have supposed $\eta > p$, hence (see Lemma 4(iii)) the limes superior of q_n/q_{n-1}^p is infinity. Let $(n_k)_k$ be an increasing sequence of natural numbers such that $\lim_k q_{n_k}/q_{n_k-1}^p = \infty$. Let $0 < \delta < 1$ and define

$$B_k = \left[\frac{q_{n_k}}{\log \frac{q_{n_k}}{q_{n_k-1}^p}} \right], \quad c_k = \left[\left(\frac{q_{n_k}}{q_{n_k-1}^p} \right)^{1-\delta} \right],$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$. Then $B_k/q_{n_k} \to 0$ and $c_k q_{n_k-1}^p / B_k \to 0$, hence (10) holds. By Theorem 2, for every F from a dense G_δ subset of $\mathcal{C}_0^p$ there exists an increasing sequence of natural numbers k_j such that

$$\lim_j \lambda(|S_{B_{k_j}}(F)| > c_{k_j}) = 1.$$

For $\epsilon, \delta > 0$, $1 - \frac{p}{\eta} - \epsilon > 0$, $q_n = q_{n_k}$ and $q = B_k$ we obtain from this and from (21) that

$$\lambda\Big(|S_q(F)| > q_n^{(1-\delta)(1-\frac{p}{\eta}-\epsilon)}\Big) > 1 - \epsilon$$

for infinitely many q, q_n, $q < q_n$. We can take $\delta > 0$ arbitrarily small, hence the third statement of Theorem 5 follows. ∎

5. Rotations with bounded partial quotients

In the case of bounded partial quotients, the results will be much more meager than in the preceding case. If α has bounded partial quotients, it is well-known that $F \in \mathcal{A}_0$ with a square integrable derivative is a coboundary. Therefore, each function from $\mathcal{L}_0$ (and hence also from $\mathcal{C}_0^k$, $1 \le k < \infty$) is a coboundary. The next result shows that for the functions with $\int |F'|^2 dx = \infty$ this need not be the case. The proof of the theorem works for any irrational number α but only the bounded partial quotients case is interesting now. The other case follows from Theorem 2 and the fact that any measurable coboundary is stochastically bounded.

THEOREM 6: *Let α have bounded partial quotients a_n. Then there exists a dense G_δ subset of $\mathcal{A}_0$ of functions which are not coboundaries.*

As we shall see in the proof, we can guarantee an existence of a dense G_δ set of functions $F \in \mathcal{A}_0$ such that for some $c_k \to \infty$, $n_k \to \infty$ (depending on F), the distributions of $(1/c_k)S_{n_k}(F)$ weakly converge to the standard normal law. At this moment, however, we are not able to prove a result as strong as Theorem 2. It is even not clear whether a distribution which is not infinitely divisible can be a weak limit point. On the other hand, the bounds for the partial sums can be found in a more satisfactory way.

THEOREM 7: *Let α be an irrational number with bounded partial quotients a_n. Then for each $F \in \mathcal{A}_0$,*

$$S_n(F) = o(\log n).$$

Moreover, for any sequence of positive numbers $(c_n)_n$ which converges to 0, there exists a dense G_δ set of $f \in \mathcal{A}_0$ for which

$$\|S_n(f)\|_\infty \geq c_n \log n$$

for infinitely many n.

The first part of Theorem 7 is well-known with several proofs; for completeness, we shall show one of them. Before proving the theorems, let us state the following general lemma:

LEMMA 5: *For $x, t \in [0,1)$ and $n = 1, 2, \dots$ let*

$$\mathbb{F}_n(x,t) = \frac{1}{q_n}\#\{i\colon 1 \leq i \leq q_n, \{i\alpha\} < x, \{q_n\{i\alpha\}\} < t\}.$$

Then

$$\lim_{n\to\infty} |\mathbb{F}_n(x,t) - x \cdot \mathbb{F}_n(1,t)| = 0.$$

Proof of the Lemma: We have $\{q_n\{i\alpha\}\} = \{iq_n\alpha\} = \{i\{q_n\alpha\}\}$. Without loss of generality we can suppose that n is even (the "odd" case is similar); then $\{q_n\alpha\} = \|q_n\alpha\| \leq 1/q_{n+1}$ so that $\{q_n\{i\alpha\}\} = i\{q_n\alpha\}$, $i = 1, \dots, q_n$. Let $x, t \in [0,1)$; if $q_n\|q_n\alpha\| < t$, then $\mathbb{F}_n(1,t) = 1$ and $\mathbb{F}_n(x,t) = \frac{1}{q_n}\#\{i\colon 1 \leq i \leq q_n, \{i\alpha\} < x\}$, so that the result follows from the uniform distribution mod 1 of the sequence $(\{k\alpha\})_k$. Now, assume $t \leq q_n\|q_n\alpha\|$ and recall that the sequence $(\{k\alpha\})_k$ is well distributed mod 1 (see [13]). Therefore $\lim_{M\to\infty} r(M)/M = 0$, where $r(M) = \sup_{j\geq 1} |x \cdot M - \#\{i\colon j \leq i \leq j + M - 1, \{i\alpha\} < x\}|$.

For M, n given put $q_n = k \cdot M + p$ with $1 \leq p \leq M - 1$. Given any $\epsilon > 0$ we can choose q_n, M and k such that $r(M)/M < \epsilon/2$, $1/k < \epsilon/2$. There exists one number j, $0 \leq j \leq k$, such that $(j \cdot M)\{q_n\alpha\} \leq t < ((j+1)M - 1)\{q_n\alpha\}$.

Therefore,

$$\begin{aligned}|x\cdot\#\{i\colon 1\le i\le q_n,\{iq_n\alpha\}<t\}-\#\{i\colon 1\le i\le q_n,\{i\alpha\}<x,\{iq_n\alpha\}<t\}|\\ \le\sum_{\ell=0}^{j-1}|x\cdot M-\#\{i\colon \ell\cdot M+1\le i\le(\ell+1)M,\{i\alpha\}<x\}|+M\\ \le k\cdot r(M)+M\le\epsilon q_n.\qquad\blacksquare\end{aligned}$$

For $k=1,2,\dots$ let d_k be a positive number, $I(k)=I=[0,\delta_k)$, ν_k be a positive integer and $n_k=\sum_{i=1}^{\nu_k}q_{\ell_i}$. The concrete values of d_k, δ_k, ν_k and ℓ_i will come out from the proof. We shall suppose that $\delta_k\ll 1/2n_k$. On $[0,1)$ we define

$$\begin{aligned}f_k(t)&=\frac{d_k}{\delta_k}\chi_I(t)-d_k,\\ \tilde F_k(t)&=\int_0^t f_k(x)\,dx,\quad F_k(t)=\tilde F_k(t)-\int_0^1\tilde F_k(x)\,dx.\end{aligned}\tag{22}$$

We thus have $Ef_k=0$, $E|f_k|=2d_k(1-\delta_k)$. We define (for a more convenient use of the Rokhlin towers we replace the former definition here)

$$S_n(f)=f+f\circ T^{-1}+\dots+f\circ T^{-n+1};$$

thus,

$$S_{n_k}(f_k)=S_{q_{\ell_1}}(f_k)+S_{q_{\ell_2}}(f_k)\circ T^{-q_{\ell_1}}+\dots+S_{q_{\ell_{\nu_k}}}(f_k)\circ T^{-q_{\ell_1}-\dots-q_{\ell_{\nu_k-1}}}$$

(cf. [17, pp. 101–102]). Let us denote

$$\begin{aligned}F_{k,j}(t)=&\int_0^t S_{q_{\ell_j}}(f_k\circ T^{-q_{\ell_1}-\dots-q_{\ell_{j-1}}})(x)\,dx\\ &-\int_0^1\int_0^u S_{q_{\ell_j}}(f_k\circ T^{-q_{\ell_1}-\dots-q_{\ell_{j-1}}})(x)\,dxdu,\\ \tilde F_{k,j}(t)=&\int_0^t S_{q_{\ell_j}}(f_k\circ T^{-1})(x)\,dx,\quad j=1,\dots,\nu_k,\ t\in[0,1).\end{aligned}$$

The functions $S_{q_{\ell_j}}(F_k)\circ T^{-q_{\ell_1}-\dots-q_{\ell_{j-1}}}$, $\tilde F_{k,j}\circ T^{1-q_{\ell_1}-\dots-q_{\ell_{j-1}}}-\int_0^1\tilde F_{k,j}(t)\,dt$ and $F_{k,j}$ have the same derivatives and zero means, hence are equal. Similarly, $S_{n_k}(F_k)=\sum_{j=1}^{\nu_k}F_{k,j}$.

In each of the intervals $[i/q_{\ell_j},(i+1)/q_{\ell_j})$, $0\le i\le q_{\ell_j}-1$, there is just one of the points $T(0),\dots,T^{q_{\ell_j}}(0)$ ([10]). The function $\tilde F_{k,j}$ is piecewise linear. In the sequel we consider the limit case where δ_k is so small that it can without loss of generality be considered equal to zero (k is considered as fixed); then $\tilde{F}_{k,j}$ has jumps of height d_k at points $T^i(0)$, $1 \le i \le q_{\ell_j}$, the derivative $\tilde{F}'_{k,j} = -d_k \cdot q_{\ell_j}$ in all other points and $\tilde{F}_{k,j}(0) = 0$. Therefore, $\tilde{F}_{k,j}(i/q_{\ell_j}) = 0$ for all $0 \le i \le q_{\ell_j} - 1$ and we denote by ξ_i the number $\{q_{\ell_j}\{u_i\alpha\}\}$ $(= q_{\ell_j}\{u_i\alpha\})$ where the integer u_i is determined by $\{u_i\alpha\} \in [i/q_{\ell_j}, (i+1)/q_{\ell_j})$ and $1 \le u_i \le q_{\ell_j}$. Finally the function $\tilde{F}_{k,j}$ is on the interval $(u_i, u_{i+1}]$ linear, decreasing with slope $-d_k \cdot q_{\ell_j}$, and its extremes are $-d_k\xi_i + d_k$ and $-d_k\xi_i$.

Proof of Theorem 6: Using the auxiliary functions $\tilde{F}_{k,j}$ we shall show that $F_{k,j}$ can be approximated by independent random variables (which will be denoted $\tilde{F}_{k,j}$). From $\tilde{F}_{k,j}(i/q_\ell) = 0$, $0 \le i \le q_{\ell_j} - 1$, the existence of jumps of height d_k at points $T^i 0$, $1 \le i \le q_{\ell_j}$, and $\tilde{F}'_{k,j} = -d_k q_{\ell,j}$ at all other points, we get

$$\frac{1}{12} d_k^2 \le \int_0^1 F_{k,j}^2(x)\,dx \le \frac{1}{3} d_k^2. \tag{23}$$

The probability that $\tilde{F}_{k,j} < t$ on $[i/q_{\ell_j}, (i+1)/q_{\ell_j})$ can be computed as

$$\begin{array}{rll} 0 & \text{for } t < 0, & \xi_i < \dfrac{|t|}{d_k}, \\ \dfrac{1}{q_{\ell_j}}\left(\xi_i - \dfrac{|t|}{d_k}\right) & \text{for } t < 0, & \xi_i \ge \dfrac{|t|}{d_k}, \\ \dfrac{1}{q_{\ell_j}}\left(\xi_i + \dfrac{t}{d_k}\right) & \text{for } 0 \le t, & \xi_i < 1 - \dfrac{t}{d_k}, \\ \dfrac{1}{q_{\ell_j}} & \text{for } 0 \le t, & \xi_i \ge 1 - \dfrac{t}{d_k}. \end{array}$$

Suppose that $x \in [0,1]$ and $x = \mu/q_{\ell_j}$ for some $\mu \in \{0, \dots, q_{\ell_j} - 1\}$. Then $\lambda([0,x) \cap \{\tilde{F}_{k,j} < t\})$ equals

$$\begin{cases} \sum\limits_{i/q_{\ell_j} < x} \frac{1}{q_{\ell_j}}(\xi_i - \frac{|t|}{d_k})\chi_{[|t|/d_k,1)}(\xi_i), & \text{for } t < 0, \\ \sum\limits_{i/q_{\ell_j} < x} \frac{1}{q_{\ell_j}}(\xi_i + \frac{t}{d_k})\chi_{[0,1-t/d_k)}(\xi_i) + \sum\limits_{i/q_{\ell_j} < x} \frac{1}{q_{\ell_j}}\chi_{[1-t/d_k,1)}(\xi_i), & \text{for } t \ge 0. \end{cases}$$

Let us suppose that t is fixed, $t < 0$, and denote

$$g(i) = \left(\xi_i - \frac{|t|}{d_k}\right)\chi_{[|t|/d_k,1)}(\xi_i), \quad 0 \le i \le q_{\ell_j} - 1.$$

We have $0 \leq g \leq 1$. Considering g as a random variable on the probability space $\{0,\dots,q_{\ell_j}-1\}$ with the measure $\mu(i) = 1/q_{\ell_j}$ and using the fact that $\int g\,d\mu = \int_0^1 \mu(g > z)\,dz = \lim_{n\to\infty}(1/n)\sum_{r=1}^n \mu(g > r/n)$ uniformly for all g with values in $[0,1)$, we get

$$\frac{1}{q_{\ell_j}} \sum_{0\leq i<x\cdot q_{\ell_j}} g(i) = \lim_{n\to\infty}(1/n)\sum_{r=1}^n \frac{1}{q_{\ell_j}} \sum_{0\leq i<x\cdot q_{\ell_j}} \chi_{[r/n,1)}(g(i))$$
$$= \lim_{n\to\infty}(1/n)\sum_{r=1}^n \frac{1}{q_{\ell_j}} \#\left\{i\colon 0\leq i<x\cdot q_{\ell_j}, \xi_i \geq \frac{|t|}{d_k}+\frac{r}{n}\right\}.$$

Let the positive integer n and $\epsilon > 0$ be fixed. From the definition of ξ_i and from Lemma 5, for ℓ_j sufficiently big we get

$$\Bigg|\frac{1}{n}\sum_{r=1}^n \frac{1}{q_{\ell_j}} \#\left\{i\colon 0\leq i<x\cdot q_{\ell_j}, \xi_i \geq \frac{|t|}{d_k}+\frac{r}{n}\right\}$$
$$- \frac{x}{n}\sum_{r=1}^n \frac{1}{q_{\ell_j}} \#\left\{i\colon 0\leq i\leq q_{\ell_j}-1, \xi_i \geq \frac{|t|}{d_k}+\frac{r}{n}\right\}\Bigg| < \epsilon.$$

For $\epsilon > 0$ and q_{ℓ_j} sufficiently big we thus have

$$|\lambda([0,x)\cap\{\tilde F_{k,j} < t\} - x\cdot\lambda(\tilde F_{k,j} < t)| < 2\epsilon. \tag{24}$$

The same result we get for $t \geq 0$.

For any $\eta > 0$ we can find finite valued functions $\hat F_{k,j}$ on $[0,1)$ such that $\|F_{k,j} - \hat F_{k,j}\|_2 < \eta$, and for each $t \in \mathbb{R}$ there exists an interval $J_t \subset \mathbb{R}$, $t \in J_t$ for $J_t \neq \emptyset$, $\lambda(\{\hat F_{k,j} = t\}) = \lambda(\{F_{k,j} \in J_t\})$, $\int_0^1 \hat F_{k,j}(x)\,dx = 0$, $j = 1,\dots,\nu_k$. From (24) and the definition of $\hat F_{k,j}$ it follows that if the numbers $\ell_{j+1} - \ell_j$ are sufficiently big,

$$\sum\Big\{|\lambda(\hat F_{k,j} = t_j | \hat F_{k,j-1} = t_{j-1},\dots,\hat F_{k,1} = t_1) - \lambda(\hat F_{k,j} = t_j)|\colon$$
$$\lambda(\hat F_{k,1} = t_1,\dots,\hat F_{k,j-1} = t_{j-1}) > 0\Big\} < \eta.$$

Therefore, for any $\epsilon > 0$ and $\ell_{j+1} - \ell_j$, $1 \leq j \leq \nu_k - 1$ sufficiently big, there exists a sequence $\bar F_{k,1},\dots,\bar F_{k,\nu_k}$ of independent random variables (functions on $[0,1)$) with

$$\int_0^1 \bar F_{k,h}(x)\,dx = 0, \tag{25}$$
$$\|\bar F_{k,j} - F_{k,j}\|_2 < \epsilon, \quad j = 1,\dots,\nu_k.$$

By (23) we have $(1/12)d_k^2 \le \int_0^1 F_{k,j}^2(x)\,dx \le (1/3)d_k^2$. For ϵ sufficiently small we can suppose $(1/20)d_k^2 \le \int_0^1 \bar{F}_{k,j}^2(x)\,dx \le (1/2)d_k^2$, so that for $\Sigma_k = \sum_{j=1}^{\nu_k} \bar{F}_{k,j}$, $\sigma_k^2 = \int_0^1 \Sigma_k^2(x)\,dx$, we have

$$\frac{\nu_k}{20} d_k^2 \le \sigma_k^2 \le \frac{\nu_k}{2} d_k^2.$$

Taking ν_k so big that $d_k^2 \nu_k \to \infty$ as $k \to \infty$ we thus get $|\bar{F}_{k,j}|/\sigma_k \to 0$ as $k \to \infty$ independently of j and by the CLT (see, e.g., [18, Theorem 13]),

$$\frac{1}{\sigma_k} \sum_{j=1}^{\nu_k} \bar{F}_{k,j} \to N(0,1) \quad \text{in distribution as } k \to \infty.$$

Letting ℓ_j increase sufficiently rapidly we can make the ϵ in (25) sufficiently small so that

$$\frac{1}{\sigma_k} \sum_{j=1}^{\nu_k} F_{k,j} = \frac{1}{\sigma_k} S_{n_k}(F_k)$$

converge to $N(0,1)$ as well.

By (22), the $\mathcal{A}_0$ norm of F_k is not greater than $2d_k$. Hence, choosing d_k so that $\sum_{k=1}^{\infty} d_k < \infty$ we get $\sum_{k=1}^{\infty} F_k = F \in \mathcal{A}_0$.

By Lemma 3, each of the functions F_k can be in $\mathcal{A}_0$ arbitrarily closely approximated by a coboundary with a transfer function in $\mathcal{A}_0$; without loss of generality, we thus can replace the F_k s by the coboundaries, hence each of the partial sums $\sum_{i=1}^{k-1} F_i$ becomes a coboundary. We can thus choose the numbers ν_k so big with respect to $\sum_{i=1}^{k-1} \nu_i$ that

$$\lim_{k\to\infty} \frac{1}{\sigma_k} \left\| S_{n_k}\Big(\sum_{i=1}^{k-1} F_i\Big) \right\|_2 = 0.$$

When $F_1, \dots, F_k$ are given, we choose $d_{k+1}, d_{k+2}, \dots$ so small that

$$\lim_{k\to\infty} \frac{1}{\sigma_k} \left\| S_{n_k}\Big(\sum_{i=k+1}^{\infty} F_i\Big) \right\|_2 = 0.$$

This way, $(1/\sigma_k)S_{n_k}(F) \to N(0,1)$ in distribution as $k \to \infty$. As $\sigma_k \to \infty$, the partial sums $S_n(F)$ cannot be stochastically bounded. By [15], F thus cannot be a coboundary.

The set of coboundaries $G - G \circ T$, $G \in \mathcal{A}_0$, is dense in $\mathcal{A}_0$. Hence, for each $N, M \in \mathbb{N}$ the set $\mathcal{H}_N(M) = \{F \in \mathcal{A}_0\colon (\exists n \ge N)(\lambda(|S_n(F)| > M) > 1/2)\}$

contains a dense and open subset of $\mathcal{A}_0$. The set $\bigcap_{N=1}^{\infty}\bigcap_{M=1}^{\infty}\mathcal{H}_N(M)=\mathcal{H}$ thus contains a dense G_δ subset of $\mathcal{A}_0$ and for each $F\in\mathcal{H}$, the sequence $S_n(F)$ is not stochastically bounded, hence F is not a coboundary. ∎

Proof of Theorem 7: First, we shall prove the second part of the Theorem. Let K be a positive integer satisfying $a_n \le K-1$ for all partial quotients a_n. Let m be a positive integer (the value will be specified later) and let k be fixed. We put $\ell_i = i\cdot m$, $i=1,\dots,\nu_k$.

Let $n=\ell_j=m\cdot j$ be a given even number (hence $\{q_n\alpha\}=\|q_n\alpha\|$), $\xi_i = \{q_n\{u_i\alpha\}\}$ where $\{u_i\alpha\}\in[i/q_n,(i+1)/q_n)$, $0\le u_i\le q_n$. Using the same argument as in the proof of Lemma 5, we have $\xi_i=\{u_i\{q_n\alpha\}\}$, hence $0\le\xi_i\le q_n\{\alpha q_n\}$ (≤ 1). Notice that by (4),

$$q_{u+2}=a_{u+2}q_{u+1}+q_u\le K\cdot q_{u+1}$$

for all integers $u\ge 0$. By (7), (5), (4) we thus get

$$\begin{aligned}1-q_n\|q_n\alpha\| &\ge 1-q_{n+1}\|q_n\alpha\| = q_n\|q_{n+1}\alpha\|\\ &\ge q_n/(2q_{n+2})\ge q_n/(2K^2q_n)=1/(2K^2),\end{aligned}$$

hence

$$-d_k\xi_i+d_k\in(0,1-1/(2K^2)],\quad 0\le i\le q_n-1$$

so that the maximum $-d_k\xi_i+d_k$ of $\tilde F_{k,j}$ on the interval $(\{u_i\alpha\},\{u_{i+1}\alpha\})$ is greater than $d_k/(2K^2)$. From the definition of the function $\tilde F_{k,j}$ we get that

$$\tilde F_{k,j}(x)\ge\frac{d_k}{4K^2}\quad\text{for } x\in\left(\{u_i\alpha\},\{u_i\alpha\}+\frac{1}{4q_nK^2}\right],$$

$i=0,\dots,q_n-1$. A similar situation happens when n is odd.

For m big enough we have $q_{u+m}\ge 12K^2q_u$ for all $u=1,2,\dots$ and $q_{\ell_1}+\dots+q_{\ell_{j-1}}<q_{\ell_j}$. Hence, there exist $x\in[0,1)$ such that

$$\tilde F_{k,j}(T^{1-q_{\ell_1}-\dots-q_{\ell_{j-1}}}x)\ge\frac{d_k}{4K^2}\quad\text{for all } j=1,2,\dots,\nu_k,\quad \ell_j=j\cdot m,$$

hence

$$S_{n_k}(F_k(x))\ge\nu_k\frac{d_k}{4K^2}.$$

From $q_{n+1}\le Kq_n$ we get

$$n_k=\sum_{j=1}^{\nu_k}q_{m\cdot j}\le\sum_{j=1}^{\nu_k}K^{m\cdot j}\le K^{m(\nu_k+1)},$$

hence

$$\frac{1}{m}\log_K n_k \le \nu_k + 1.$$

Therefore,

$$S_{n_k}(F_k(x)) \ge \frac{d_k}{4K^2 \cdot m}(\log_K n_k - m).$$

We can choose ν_k as big as we need, hence we can get

$$S_{n_k}(F_k(x)) > 3c_{n_k}\log_2 n_k.$$

The functions F_k can be in $\mathcal{A}_0$ arbitrarily closely approximated by coboundaries with transfer functions in $\mathcal{A}_0$ (see Lemma 3); we can thus replace them this way. Having the numbers $n_1, \dots, n_{k-1}$ fixed, we can therefore choose n_k so big that

$$\sum_{j=1}^{k-1} \big|\sup S_{n_k}(F_j)\big|/(c_{n_k}\log_2 n_k) \le \frac{1}{2};$$

choosing the number d_k sufficiently small we get

$$\sum_{j=k+1}^{\infty} \big|\sup S_{n_k}(F_j)\big|/(c_{n_k}\log_2 n_k) < \frac{1}{2},$$

and

$$F = \sum_{k=1}^{\infty} F_k$$

converging in $\mathcal{A}_0$. We have

$$\sup S_{n_k}(F)/(c_{n_k}\log_2 n_k) > 2$$

for infinitely many integers k.

For any $G \in \mathcal{A}_0$ we have

$$S_{n_k}(F + G - G\circ T)/(c_{n_k}\log_2 n_k) > 1$$

for infinitely many integers k. The set of coboundaries $G - G\circ T$, $G \in \mathcal{A}_0$, is dense in $\mathcal{A}_0$ (see Lemma 3). The set $\mathcal{H}_n = \{F\colon \exists N \ge n, \sup S_N(F) > c_N \log_2 N\}$ is dense and open in $\mathcal{A}_0$ and for each $n = 1, 2, \dots$, $\mathcal{H} = \bigcap_{n=1}^{\infty} \mathcal{H}_n$ is a dense G_δ subset of $\mathcal{A}_0$ and for each $F \in \mathcal{H}$, $S_n(F) > c_n \log_2 n$ infinitely many times. This proves the second part of Theorem 7.

Let us suppose that for some $c > 0$ and $F \in \mathcal{A}_0$ we have

$$\limsup_{n\to\infty} \sup S_n(F)/\log_2 n > c. \tag{26}$$

For any bounded function G we then have

$$\limsup_{n\to\infty} \sup S_n(F + G - G \circ T)/\log_2 n > c.$$

By Lemma 3 the set of coboundaries $G - G \circ T$ with $G \in \mathcal{A}_0$ is dense in $\mathcal{A}_0$, hence for every $k > 0$, in every open nonempty subset of $\mathcal{A}_0$ there exists a function $G - G \circ T + k \cdot F$, $G \in \mathcal{A}_0$. For any $N \in \mathbb{N}$ there exists $n \geq N$ with

$$\sup S_n(k \cdot F + G - G \circ T)/\log_2 n > k \cdot c.$$

This property remains valid for a sufficiently small open neighborhood of $G - G \circ T + k \cdot F$, hence the set

$$H_{N,k} = \{F \in \mathcal{A}_0 \colon \exists\, n \geq N,\ \sup S_n(F)/\log_2 n > k \cdot c\}$$

is open and dense in $\mathcal{A}_0$. Then $H = \bigcap_{N=1}^{\infty} \bigcap_{k=1}^{\infty} H_{N,k}$ is a dense G_δ subset of $\mathcal{A}_0$, hence nonempty. For each $F \in H$ we have

$$\limsup_{n\to\infty} \sup S_n(F)/\log_2 n = \infty.$$

The discrepancy ND_N^* of the sequence $\{n\alpha\}$ is bounded by $d \cdot \log_2 N$ for some constant d (see [13], Theorem 3.4, p.125) and by the Denjoy–Koksma's theorem, $|S_n(F)| \leq b \cdot \log_2 n$ for all $n = 1, 2, \ldots$ and $b = d \cdot V(F)$ where $V(F)$ denotes the total variation of F. Therefore, $\limsup_{n\to\infty} \sup |S_n(F)|/\log_2 n$ must be finite. This contradiction shows that (26) cannot hold for any $F \in \mathcal{A}_0$, hence

$$\limsup_{n\to\infty} \sup |S_n(F)|/\log_2 n = 0. \qquad \blacksquare$$

Remark: The proof of the existence of F_k such that

$$S_{n_k}(F(x)) \geq \frac{d_k \nu_k}{4K^2}$$

can be easily extended to a function F which has a jump of height d at $1/2$ and is linear otherwise (with constant slope and $F(0) = 0$). F is a zero mean bounded variation function and, using the same approach as in the preceding proof, we can show that there exists a discontinuum of points x for which

$$S_{n_k}(F(x)) \geq \frac{d \cdot \nu_k}{4K^2}$$

for all integers ν_k, $n_k = \sum_{i=1}^{\nu_k} q_{i \cdot m}$. Therefore,

$$\sup S_n(F) > c \cdot \log_2 n$$

for some $c > 0$ and infinitely many n. The proof of the first part of Theorem 7 thus cannot work for general bounded variation functions, i.e. the coboundaries $G - G \circ T$ with G bounded are not dense in that space (with respect to the variation topology). In fact, even the coboundaries with measurable transfer functions are not dense in that space.

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